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ON STABILIZATION OF DISCRETE MONOTONE DYNAMICAL SYSTEMS

BY

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ABSTRACT

A stabilization theorem for discrete strongly monotone and nonexpansive dynamical systems on a Banach lattice is proved. This result is applied to a periodic-parabolic semilinear initial-boundary value problem to show the convergence of solutions towards periodic solutions.

1. Introduction

In this paper we are concerned with discrete-time monotone dynamical systems $(S^n)_{n\in\mathbf{N}}$ on a Banach lattice X. The point of departure in our study is the construction for such systems of a nontrivial semicontinuous Liapunov operator $V: X \to X$. Specializing further to nonexpansive maps S we establish that the ω-limit set corresponding to a relatively compact positive orbit is a subset of a special level set of the operator, $\{x \in X : V(x) = q\}$, where q is a rest point of S. Thus we obtain in this setting a variant of the La Salle Invariance Principle [L]. We conclude that for discrete-time strongly monotone nonexpansive dynamical systems the ω-limit set of relatively compact orbits is a single point. This result is well suited to prove stabilization of solutions of T-periodic initial-boundary value problems towards a T-periodic solution (whose existence we need not to assume *a priori*). In this context S is the section map. We illustrate this point by giving an application to a class of

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parabolic equations and so in particular extend to the periodic case a result of Hirsch [Hi 1].

Our method is closely related to that used in Alikakos–Bates [A–B], where the continuous-time analog of our results is established. Dafermos [D 2] was the first to utilize semicontinuous Liapunov functions in his studies on the asymptotic behaviour. Here we follow a number of his ideas. Other related work on discrete-time monotone systems can be found in Hirsch [Hi 2]. For the behaviour of the iterates of a nonexpansive mapping without monotonicity hypotheses we refer to Pazy [P] and Brézis [B] where further references can be found.

Making more intensive use of results of Dafermos [D 2,3] we plan to extend in [A–H] our results to the class of strongly monotone uniform processes that in particular apply to almost periodic parabolic differential equations.

2. Statement of the main results

Let X be a Banach lattice with the properties

(X.1) X is σ-order-complete: the supremum of countable majorized subsets always exists,

(X.2) X has a σ-order-continuous norm: any increasing sequence with a supremum is convergent.

Let $P = X_+$ be the positive cone in X which defines the ordering. For some of the results in this paper we will require X to satisfy in addition

(X.3) there exists a Banach space $Z \subset X$ such that $P \cap Z$ has nonempty interior in Z,

(X.4) X is strictly convex.

We denote by $\leqq$ the order relation that P induces, and write $x < y$ if $y - x \in P$, $x \neq y$, as well as $x \ll y$ if $y - x \in \operatorname{int}(P \cap Z)$.

Let $Y \subset X$ be a closed subset and $S: Y \to Y$. We say that S is *monotone* if $x \leqq y$ implies that $S(x) \leqq S(y)$, and *strongly monotone* provided $S(Y) \subset Z$ and $x < y$ implies $S(x) \ll S(y)$. Following [M] we introduce

DEFINITION 2.1. An element $\bar{u} \in Y$ is called a *supersolution* if $S(\bar{u}) \leqq \bar{u}$, and $\underline{u} \in Y$ is a *subsolution* if $\underline{u} \leqq S(\underline{u})$.

We remark that for monotone S the infimum of two supersolutions is again a supersolution, and the supremum of two subsolutions again a subsolution.

DEFINITION 2.2. A point y is a positive limit of $S^n(x)$ if there is a sequence of integers n_i such that $n_i \to +\infty$ and $S^{n_i}(x) \to y$ in X as $i \to \infty$. The positive limit set $\omega(x)$ of x is the set of all its positive limit points. Relative to S a set $H \subset Y$ is positively invariant if $S(H) \subset H$, and invariant if $S(H) = H$.

If S is continuous on Y, every positive limit set is closed and positively invariant. If the positive orbit $\gamma^+(x) := \{S^n(x) : n \in \mathbf{N}\}$ is relatively compact in X, then $\omega(x)$ is nonempty, compact, invariant and invariantly connected (i.e. it is not the union of two nonempty disjoint closed invariant sets), and it is the smallest closed set that $S^n(x)$ approaches as $n \to \infty$ ([L, Thm. 5.2, p. 4]).

In the rest of this paper we take as Y the order interval $[\underline{u}, \bar{u}]$ defined by order-related sub- and supersolutions $\underline{u} < \bar{u}$. It follows from the monotonicity of S that $S(Y) \subset Y$.

Now we are ready to state our main result.

THEOREM 1. *Let X satisfy (X.1)–(X.4), and let $S : Y \to Y$ be a strongly monotone map which is nonexpansive: $\| S(x) - S(y) \|_X \leqq \| x - y \|_X$ for $x, y \in Y$. Let $u_0 \in Y$, and assume that $\gamma^+(u_0)$ is relatively compact in X. Then $\omega(u_0) = \{v\}$ for some $v \in Y$ (which in general depends on u_0).*

As a consequence of the positive invariance of $\omega(u_0)$, v is a rest point, i.e. $S(v) = v$.

REMARK 2.3. As a simple application of Theorem 1 we obtain a continuous-time analog of this result that was established in [Hi 1] (see also [A–B]). Let $(S(t))_{t \geqq 0}$ be a continuous semigroup of nonexpansive, monotone maps in X. We define $\underline{u}(\bar{u})$ to be a subsolution (supersolution) if $S(t)\underline{u} \geqq \underline{u}(S(t)\bar{u} \leqq \bar{u})$ for $t > 0$. Thus the order-interval $Y = [\underline{u}, \bar{u}]$ determined by order-related sub- and supersolutions $\underline{u} < \bar{u}$ is positively invariant under S. Let X satisfy the hypotheses (X.1)–(X.4). Assume that for $t > 0$, $S(t)Y \subset Z$ and the map $S(t)$ is strongly monotone. For $u_0 \in X$ we define the positive orbit $\gamma^+(u_0) := \{S(t)u_0 : t \geqq 0\}$ and the ω-limit set $\omega(u_0) := \{\xi \in X : \xi = \lim S(t_n)u_0$ for some sequence $t_n \to +\infty\}$.

THEOREM 1′. *Let X, $(S(t))_{t \geqq 0}$ be as above, $u_0 \in Y$, and assume that $\gamma^+(u_0)$ is relatively compact in X. Then $\omega(u_0) = \{v\}$, where v is a rest point of S, i.e. $S(t)v = v$ for $t \geqq 0$.*

PROOF. Let $T > 0$ be an arbitrary number and define $S_T(u) := S(T)u$ for $u \in Y$. It follows that $(S_T^n)_{n \in \mathbf{N}}$ is a discrete-time dynamical system satisfying the hypotheses of Theorem 1. Thus, for $n \to \infty$, (2.1) $$S_T^n(u_0) \to v, \quad S_T(v) = v.$$

Next define $S_{T/i}(u) := S(T/i)u$ to obtain that $S_{T/i}^n(u_0) \to v$, $S_{T/i}(v) = v$, $i = 1, 2, \ldots$. It follows from the semigroup property and the continuity of the map $t \to S(t)u_0$ that $S(t)v = v$ for all $t > 0$, and since S is nonexpansive, $\omega(u_0) = \{v\}$. □

We now introduce the setting for the application of Theorem 1 to periodic-parabolic differential equations. Let Ω be a bounded domain in $\mathbf{R}^N$ ($N \geqq 1$) with boundary $\partial\Omega$ of class C^2, and let $\mathcal{L} := \partial/\partial t + \mathcal{A}(x, D)$ be a uniformly parabolic linear differential expression with

$$\mathcal{A}(x, D)u = -\sum_{j,k=1}^{N} D_j(a_{jk}(x)D_k u) + \sum_{j=1}^{N} a_j(x)D_j u + a_0(x)u$$

($D_j = \partial/\partial x_j$). We assume that the coefficient functions $a_{jk} = a_{kj}$ and a_j belong to $C_1(\overline{\Omega})$ and that $a_0 \in C(\overline{\Omega})$. Let further $\beta \in C^1(\partial\Omega, \mathbf{R}^N)$ be an outward pointing, nowhere tangent vectorfield on $\partial\Omega$ and $\beta_0 \in C^1(\partial\Omega)$, $\beta_0 \geqq 0$. Define the boundary operator $\mathcal{B} = \mathcal{B}(x, D)$ either by $\mathcal{B}u = u$ (Dirichlet case) or by $\mathcal{B}u = \partial u/\partial\beta + \beta_0 u$ (Neumann or regular oblique derivative boundary conditions). Finally let the continuous function $g : (x, t, \xi) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R} \to g(x, t, \xi) \in \mathbf{R}$ be Lipschitz continuous in (x, ξ) and Hölder continuous and T-periodic in t, for some given $T > 0$. Assume that $\partial g/\partial\xi$ exists, and that it enjoys the same properties.

We consider the semilinear parabolic initial-boundary value problem

(2.2) $$\begin{cases} \mathcal{L}u = g(x, t, u) & \text{in } \Omega \times \mathbf{R}^+, \\ \mathcal{B}u = 0 & \text{in } \partial\Omega \times \mathbf{R}^+, \\ u(\cdot, 0) = u_0 & \text{on } \overline{\Omega}. \end{cases}$$

Definition 2.4. The function $\bar{u} \in C^{2,1}(\overline{\Omega} \times]0, T]) \cap C^{1,0}(\overline{\Omega} \times [0, T])$ is called a *supersolution* on the interval $[0, T]$ provided

$$\begin{cases} \mathcal{L}\bar{u} \geqq g(\cdot, \cdot, \bar{u}) & \text{in } \Omega \times]0, T], \\ \mathcal{B}\bar{u} \geqq 0 & \text{on } \partial\Omega \times]0, T]; \end{cases}$$

a subsolution is defined with reversed inequality signs.

A simple consequence of Theorem 1 is the following

THEOREM 2. *Suppose* $\underline{u} < \bar{u}$ *are sub- and supersolutions on* $[0, T]$, *respectively, in the sense of Definition* 2.4, *with*

$$\underline{u}(\cdot, 0) \leqq \underline{u}(\cdot, T), \quad \bar{u}(\cdot, 0) \geqq \bar{u}(\cdot, T) \qquad \text{on } \bar{\Omega}. \tag{2.3}$$

Let

$$\bar{g}(x) := \max \frac{\partial g}{\partial \xi}(x, t, \xi),$$

where the maximum is taken over the set $\{(t, \xi) \in [0, T] \times \mathbf{R} : \underline{u}(x, t) \leqq \xi \leqq \bar{u}(x, t)\}$, *and assume that the pair* $(-\bar{\mathscr{A}}, \mathscr{B})$, *where* $\bar{\mathscr{A}} := \mathscr{A} - \bar{g}(x)$, *generates a contraction semigroup on* $X = L^2(\Omega)$. *Then the solution of* (2.2) *with* $u_0 \in [\underline{u}(\cdot, 0), \bar{u}(\cdot, 0)]$ *exists for all* $t \geqq 0$ *and converges strongly in* X *towards a* T*-periodic solution* w *of*

$$\begin{cases} \mathscr{L} w = g(\cdot, \cdot, w) & \text{in } \Omega \times \mathbf{R} \\ \mathscr{B} w = 0 & \text{on } \partial\Omega \times \mathbf{R} \end{cases}$$

as $t \to +\infty$.

Sufficient conditions for the positive analytic semigroup generated by $(-\bar{\mathscr{A}}, \mathscr{B})$ in $X = L^2(\Omega)$ to be nonexpansive, i.e. for the realization $\bar{A}_2$ in X to be a monotone operator, can be found in [Am, Sect. 11]. For example, let $\beta = \beta_a$ be the conormal vectorfield on $\partial\Omega$, let $\mathscr{B} = \partial/\partial\beta_a$, and assume that

$$a_0 - \bar{g} - \sum_{j=1}^{N} D_j a_j \geqq 0 \quad \text{in } \Omega, \qquad \sum_{j=1}^{N} a_j \nu_j \geqq 0 \quad \text{on } \partial\Omega$$

($\nu = (\nu_1, \ldots, \nu_N)$ is the outer normal to $\partial\Omega$).

Theorem 2 is an extension to the periodic case of a result due to Hirsch [Hi 1] (see also [*A–B*]). We note in passing that the hypotheses in [Hi 1,A–B] do not seem to suffice for the generated semigroup to be nonexpansive in X. We mention also that if $\bar{A}_2$ is a nonnegative *selfadjoint* operator, Theorem 2 follows from Theorem 3 (ii) of Kenmochi-Ôtani [K–O]. In the nonvariational case, no such result appears to be known so far.

3. An invariance principle for time-discrete monotone and nonexpansive dynamical systems

Let $S: Y \to Y$ be monotone and continuous, $Y = [\underline{u}, \bar{u}] \subset X$, and assume that X satisfies conditions $(X.1)$–$(X.2)$. For $u \in Y$ we set $\Sigma_u := \{\phi \in Y : \phi \geqq u$, ϕ supersolution$\}$. Note that Σ_u is nonempty.

LEMMA 3.1. *Σ_u possesses a unique minimal element denoted by $\bar{V}(u)$.*

The map $\bar{V}: Y \to Y$ is called the *upper Liapunov operator*.

PROOF OF LEMMA 3.1. Let J be a strictly positive linear functional on X. We construct a sequence $(\phi_n)_{n \in \mathbf{N}}$ in Σ_u as follows:

ϕ_0 is arbitrary;

$\phi_{n+1} \leqq \phi_n$ is such that $J(\phi_n - \phi_{n+1}) \geqq \frac{1}{2} m(\phi_n)$ where

$$m(\phi) := \sup\{J(\phi - \psi) : \psi \in \Sigma_u, \psi \leqq \phi\}, \phi \in \Sigma_u.$$

Since $J(\phi_n - \phi_{n+1}) + J(\phi_{n+1} - \psi) = J(\phi_n - \psi)$ $(\psi \in \Sigma_u, \psi \leqq \phi_{n+1})$, we have

$$\tfrac{1}{2} m(\phi_n) + m(\phi_{n+1}) \leqq m(\phi_n)$$

and infer that $m(\phi_n) \to 0$ as $n \to \infty$. Set $\phi_\infty := \lim_{n \to \infty} \phi_n$; by $(X.1)$–$(X.2)$ this limit exists and lies in Y. It follows from $S(\phi_n) \leqq \phi_n$ by continuity that ϕ_∞ is a supersolution. Hence $\phi_\infty \in \Sigma_u$. We claim that ϕ_∞ is a minimal element of Σ_u. Indeed, assume there exists $\psi \in \Sigma_u$ with $\psi < \phi_\infty$. Then $J(\phi_\infty - \psi) > 0$, contradicting that

$$J(\phi_\infty - \psi) \leqq J(\phi_n - \psi) \leqq m(\phi_n) \to 0 \qquad (n \to \infty).$$

The uniqueness of a minimal element of Σ_u is a consequence of the fact that Σ_u is closed under the inf operation. □

Analogously we construct the maximal element $\underline{V}(u)$ of the set of subsolutions lying below u.

The *technique* used in the construction of $\bar{V}(u)$ is from the unpublished work [C–F–P].

LEMMA 3.2. *$\bar{V}$ is nonincreasing along trajectories, i.e. for $u_0 \in Y$, $\bar{V}(S^{n+1}(u_0)) \leqq \bar{V}(S^n(u_0))$, $n \in \mathbf{N}$.*

PROOF. By definition, $S^n(u_0) \leqq \bar{V}(S^n(u_0))$. Hence by monotonicity $S^{n+1}(u_0) \leqq S(\bar{V}(S^n(u_0))) \leqq \bar{V}(S^n(u_0))$ since $\bar{V}(S^n(u_0))$ is a supersolution. It follows that

$$\bar{V}(S^{n+1}(u_0)) \leqq \bar{V}[\bar{V}(S^n(u_0))] = \bar{V}(S^n(u_0)). \qquad \square$$

PROPOSITION 3.3. (*Invariance principle*). *Let $S: Y \to Y$ be monotone and nonexpansive and let X satisfy $(X.1)$–$(X.2)$. Let $u_0 \in Y$, and assume that $\gamma^+(u_0)$ is relatively compact in X. Then*

$$\omega(u_0) \subset \{x \in Y : \bar{V}(x) = \bar{q}\}$$

where $\bar{q}$ is a rest point of $S: S(\bar{q}) = \bar{q}$.

PROOF. (i) Let $v \in \omega(u_0)$. We claim that $\omega(v) = \omega(u_0)$. It suffices to show that $\omega(u_0) \subset \omega(v)$ since the other inclusion follows by the positive invariance of ω. Let $w \in \omega(u_0)$. Then

$$v = \lim S^{k_n}(u_0), \quad w = \lim S^{l_n}(u_0),$$

where $k_n \to \infty$, $l_n \to \infty$. Without loss of generality we may assume that $m_n := l_n - k_n \to +\infty$. Now

$$\begin{aligned} \| S^{m_n}(v) - w \| &\leqq \| S^{m_n}(v) - S^{m_n}(S^{k_n}(u_0)) \| + \| S^{m_n}(S^{k_n}(u_0)) - w \| \\ &\leqq \| v - S^{k_n}(u_0) \| + \| S^{l_n}(u_0) - w \| \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

and hence $w \in \omega(v)$. Thus the claim is established.

(ii) We show that $\omega(u_0)$ is a subset of a level set of $\bar{V}$. Let $v, w \in \omega(u_0)$. By (i), $\omega(v) = \omega(w) = \omega(u_0)$, hence in particular $w \in \omega(v)$. Thus there exists a sequence $(k_n)_{n \in \mathbf{N}}$ with $k_n \to \infty$ such that $S^{k_n}(v) \to w$. By Lemma 3.2 the sequence of supersolutions $(\bar{V}(S^{k_n}(v)))_{n \in \mathbf{N}}$ is nonincreasing and by $(X.1)$–$(X.2)$ the limit $\lim \bar{V}(S^{k_n}(v))$ exists. By continuity of S, it is a supersolution. Since $\bar{V}(S^{k_n}(v)) \geqq S^{k_n}(v)$, it follows that $\lim \bar{V}(S^{k_n}(v)) \geqq w$, and thus $\lim \bar{V}(S^{k_n}(v)) \geqq \bar{V}(w)$. Putting things together,

$$\bar{V}(v) \geqq \bar{V}(S^{k_n}(v)) \geqq \lim \bar{V}(S^{k_n}(v)) \geqq \bar{V}(w).$$

By symmetry, $\bar{V}(v) = \bar{V}(w) =: \bar{q}$.

(iii) We claim that $\bar{q}$ is a rest point of S. Let $v \in \omega(u_0)$. Then $\bar{q} = \bar{V}(v) \geqq v$. If $\bar{q}$ is not an equilibrium, it is a strict supersolution, hence $S(\bar{q}) < \bar{q}$. Consequently $\bar{V}(S(v)) \leqq \bar{V}(S(\bar{q})) = S(\bar{q}) < \bar{q}$, since $S(\bar{q})$ is again a supersolution. But $S(v) \in \omega(u_0)$, and we arrive at a contradition to assertion (ii). $\square$

We remark that by step (i) in the above proof, if $\omega(u_0)$ contains a rest point v, then $\omega(u_0)=\{v\}$.

4. Proof of Theorem 1

By the last remark it suffices to show that $\omega(u_0)$ contains a rest point. Let $v\in\omega(u_0)$. Since $S^n(v)\in\omega(u_0)$ by positive invariance, we have

$$\bar{V}(S^n(v))=\bar{q},\quad \underline{V}(S^n(v))=\underline{q}\qquad (n\in\mathbf{N}) \tag{4.1}$$

where $\bar{q}$ and $\underline{q}$ are the rest points which Proposition 3.3 and its analog for subsolutions provide. Thus $\underline{q}\leqq S^n(v)\leqq\bar{q}$. Assume that v is not a rest point. Then $\underline{q}<v<\bar{q}$, and by strong monotonicity $\underline{q}\ll S(v)\ll\bar{q}$. Since S is nonexpansive on the closed convex subset Y of the strictly convex Banach space X, the set of fixed points of S is convex ([Br, Thm. 8.2]). Thus for each $\lambda\in[0,1]$, $q_\lambda:=\lambda\underline{q}+(1-\lambda)\bar{q}$ is again a rest point. For $0<\lambda$ sufficiently small, $S(v)<q_\lambda<\bar{q}$. Therefore $\bar{V}(S(v))\leqq q_\lambda$, which contradicts (4.1). Thus v is a rest point. □

5. Proof of Theorem 2

Set $X:=L^2(\Omega)$ and $Z:=C_0^1(\bar{\Omega})$ in case of Dirichlet boundary conditions, $Z:=C(\bar{\Omega})$ in case of Neumann or regular oblique derivative boundary conditions. Equation (2.2) can be written abstractly in the form

$$\begin{cases} du/dt=-A_2u+g(t,u), & t>0,\\ u(0)=u_0, \end{cases} \tag{5.1}$$

where A_2 is the realization of $(\mathscr{A},\mathscr{B})$ in $L^2(\Omega)$. We take as $Y:=\{u_0\in X: \underline{u}(0)\leqq u_0\leqq\bar{u}(0)\}$. By applying the maximum principle to (2.2) we obtain that for $u_0\in Y$ the solution $u(t)$ satisfies $\underline{u}(t)\leqq u(t)\leqq\bar{u}(t)$, $t\geqq 0$. Since the functions $\underline{u}(\cdot,t)$ and $\bar{u}(\cdot,t)$ are uniformly bounded in X, without loss of generality we may take g to be bounded on $\mathbf{R}^+\times X$. It follows that (5.1) is well-posed in Y, and that the set $\{u(t): t\geqq 0\}$ is relatively compact in X ([He, Paragraph 3.3]). Let $v(t)$ be another solution corresponding to some $v_0\in Y$. Then (2.2) implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)-v(t)\|_X^2 &\leqq -\int_\Omega A_2(u-v)(t)\cdot(u-v)(t)dx+\int_\Omega \bar{g}(u-v)^2(t)dx\\ &= \int_\Omega -\bar{A}_2(u-v)(t)\cdot(u-v)(t)dx,\end{aligned} \tag{5.2}$$

and by the contraction hypothesis on $\bar{\mathscr{A}}$ we obtain that

$$\|u(t_2)-v(t_2)\|_X \leqq \|u(t_1)-v(t_1)\|_X, \qquad t_1<t_2. \tag{5.3}$$

Next we define

$$S(u_0):=u(T).$$

It follows from (2.3) and the maximum principle that $S(\underline{u}(0))\geqq \underline{u}(0)$, $S(\bar{u}(0))\leqq \bar{u}(0)$, that is, $\underline{u}(0)$ and $\bar{u}(0)$ are a sub- and a supersolution for S in the sense of Definition 2.1. Finally applying the maximum principle once more we obtain that S is nonexpansive, and so Theorem 1 applies to give that

$$S^n(u_0)\to q, \quad S(q)=q. \tag{5.4}$$

Let w be the solution to (5.1) with $w(0)=q$. Uniqueness shows that w is periodic. Let $[t]$ be the number of multiples of T contained in $t>0$. Then

$$\|u(t)-w(t)\|_X \leqq \|u([t]T)-w([t]T)\|_X = \|S^{[t]}(u_0)-q\|_X \to 0$$

as $t\to+\infty$. □

We remark that we did not have to assume the existence of a periodic solution of (2.2) between $\underline{u}(t)$ and $\bar{u}(t)$. In the present situation existence is, however, well-known, even for considerably more general operators (e.g. [D–H]).

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